

## A Dynamic Model of a Self-Vibration Cycle in a Stirling Engine with Opposed Cylinders

M. Ya. Izrailovich, A. V. Sinev, V. F. Shcherbakov, and R. V. Kangun

Received September 13, 2006; in final form, February 5, 2007

**Abstract**—A dynamic model of the self-vibration cycle in an engine with opposed cylinders and two pistons located on a common guide bar is studied. To each cylinder containing a working liquid, a pair of hydraulic accumulators is attached the upper part of which contains a gas heated in one hydraulic accumulator and cooled in the other. Each hydraulic accumulator contains a membrane separating the gas and the liquid. In this way, a pressure difference is created in the cylinders that acts on each of the pistons and moves them along with the guiding bar.

**DOI:** 10.3103/S1052618807030028

At the current stage of progress in the machine-building industry, development of cyclic engines with external heating (Stirling-type engines) is becoming increasingly important. The operation principle and a description of the thermodynamic characteristics of the Stirling cycle are presented in detail in [1–3]. However, in these and other studies by Russian and foreign researchers, analysis is performed on the basis of the thermodynamic cycle alone, i.e., without taking into account the forces of inertia and resistance that act on the moving parts of the engine mechanism. Neglect of these factors results in a significant decrease in the accuracy of calculations involved in the analysis of motion and sometimes even to qualitative distortions of the ideal cycle. In [4], based on the combined analysis of the thermodynamic process and the methods of machine dynamics applied as an example to the Stirling free-piston engine, it was shown that the isochoric parts of the ideal cycle [1–3] may not be implemented. In [5], a similar analysis was made, taking into account the forward-and-backward motion of a piston and rotation of a working shaft driven by the piston via a crank-and-rod mechanism, for the Stirling engine with a crosshead mechanism.

Let us consider a model dynamic of an engine with opposed cylinders [6] (Fig. 1), where 1 is the first hydraulic accumulator (heating), 2 is the regenerator of the first hydraulic accumulator, 3 is the second hydraulic accumulator (cooling), 4 is the third hydraulic accumulator (cooling), 5 is the regenerator of the second hydraulic accumulator, 6 is the fourth hydraulic accumulator (heating), 7 is the right hydraulic cylinder, 8 is the right piston, 9 is the driving mechanism, 10 is the guiding bar, 11 is the left piston, and 12 is the left hydraulic cylinder.

The engine's working chambers consist of the working chambers of cylinders 7 and 12 with guiding bars connected in opposite directions. The double-acting cylinder is characterized by different efficient areas of the piston and piston-rod (left and right) ends. On the piping of piston ends, heaters (heat exchangers) are installed and the piping of the piston-rod ends bears coolers (cooling radiators). In the middle part of the piping connecting the ends, regenerators 2 and 5 are installed. To create an excessive pressure, gas is pumped into the working chambers through filling valves. The periodic motion of the pistons is guided by drive mechanism 9. The efficient areas of piston and piston-rod ends are pairwise equal. Owing to this, it is possible to maintain overpressure when the working chambers of an engine are filled. During the filling operation, the pressure values in the piston and piston-rod ends are the same, since the chambers are connected with piping and the hydraulic resistance of the heater, cooler, regenerator, and connecting piping is insignificant, so that the excessive pressure during filling does not create any unbalanced forces. The unbalanced forces are created during the engine operation owing to the difference of pressures in cylinders 7 and 12 operating in opposite phases.

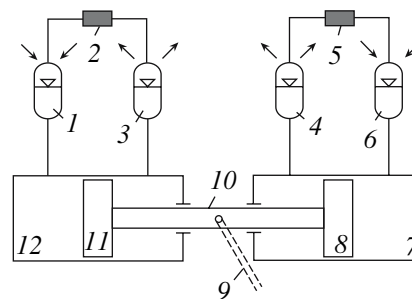


Fig. 1

The opposed location of the cylinders makes it possible to implement a positive feedback between the cylinders' working processes. This circumstance ensures a unique pattern of the engine's periodic processes without introducing any adjustment devices [7] and enables obtaining of a significant initial pressure in the working chambers. When the flywheel of the driving mechanism rotates clockwise, the volume of the piston chamber in cylinder 7 decreases and that in cylinder 12 increases. This change causes motion of the working medium along the circulation contour; the major part of the gas is heated by the heater and the total pressure in cylinder 7 increases and in cylinder 12 decreases. This process creates an increasing force acting in the direction from the right-hand side of the wheel toward its left-hand side (positive feedback between the working processes in the chambers) to the pistons' dead center. The flywheel mechanically passes the dead center and causes motion in the opposite direction. One can see that continuous motion of the wheel is maintained in absence of any valves. The working chambers in this design do not need to be air tightened, since the pressure values in the left-hand and right-hand chambers of the hydraulic cylinders are pairwise equal and the pistons are installed to maintain circulation of the working medium. In this case, only rods of hydraulic cylinders have to be tightened.

Here, we present a rigorous and a simplified model of the dynamics of the analyzed engine model.

The rigorous model enables calculation of steady-state and transient regimes by numerical methods (using the MathCAD software package). The approximate model enables derivation of closed analytical solutions for the steady-state regimes of the engine's operations.

When developing the rigorous and approximate models, the following assumptions are made: deformation of the membranes in the hydraulic accumulators when the liquid is displaced with the gas or vice versa is approximated by their displacement upwards or downwards, respectively; the working liquid is ideal and incompressible; and pressures  $p_1, p_2, p_3$ , and  $p_4$  developed in the upper (gas filled) parts of the accumulator volumes are passed directly to the working surfaces of the pistons.

By virtue of the first assumption, the membrane surfaces are assumed to be flat.

Let us introduce the following notations:  $\bar{V}_1, \bar{V}_2, \bar{V}_3$ , and  $\bar{V}_4$  are the total volumes of the liquid contained in the left-hand and right-hand chambers of the hydraulic cylinders and the lower chambers of the hydraulic accumulators;  $V_{1A}, V_{2A}, V_{3A}$ , and  $V_{4A}$  are the current values of the volume of the fluid contained in the lower chambers of the hydraulic accumulators;  $V_{1c}, V_{2c}, V_{3c}$ , and  $V_{4c}$  are the current values of the volume of the fluid contained in the left- and right-hand chambers of the hydraulic cylinders;  $S_A$  is the area of the hydraulic accumulator's membrane;  $S_c$  is the area of the hydraulic cylinder's piston;  $h$  is the thickness of the hydraulic cylinder's piston;  $l$  is the height of the chambers of hydraulic accumulators;  $H$  is the length of the chambers of hydraulic accumulators;  $x$  is the current value of the piston displacement measured from the left end of each hydraulic cylinder;  $m_{ps}$  is the mass of each piston;  $m_{rd}$  is the rod mass;  $m_l$  is the total mass of the engine's moving parts;  $y_1, y_2, y_3$ , and  $y_4$  are the current values of the liquid level in each hydraulic accumulator;  $r_1$  and  $r_2$  are the regenerator parameters characterizing the rate with which the gas mass changes as a function of the pressure difference;  $T_1$  and  $T_2$  are the temperature of heating and cooling of hydraulic accumulators, respectively;  $m_1, m_2, m_3$ , and  $m_4$  are the current values of the gas mass in each of the four hydraulic accumulators;  $m_d$  is the invariable mass of the gas contained in the "dead volume" (heater, regenerator, and cooler); and  $m_{01}$  and  $m_{02}$  are the total masses of the gas contained in the first and second and in the third and fourth hydraulic accumulators.

The balance of volumes for the first hydraulic accumulator and the left-hand side of the first hydraulic cylinder has the form

$$V_{1A} + V_{1c} = y_1 S_A + x S_c = \bar{V}_1. \quad (1)$$

From (1), it follows that

$$y_1 = (\bar{V}_1 - x S_c) / S_A. \quad (2)$$

In accordance with (2), the volume of the gas contained in the upper chamber of the first hydraulic accumulator may be found as

$$v_1 = (l - y_1) S_A = l S_A - \bar{V}_1 + x S_c. \quad (3)$$

Since  $v_1 > 0$ , formula (3) yields the boundary of the leftmost position of the piston,  $x > x_{lm} = (\bar{V}_1 - lS_A)/S_c$ . The Mendelev–Klaiperon equation for volume  $v_1$  has the form

$$p_1 v_1 = \frac{m_1}{M} R_\mu T_1, \quad (4)$$

where  $R_\mu$  is the specific gas constant and  $M$  is the molar mass of the working medium.

Formulas (3) and (4) determine the current values of the pressure  $p_1(x, m_1)$  acting from the left on the piston in the first cylinder,

$$p_1(x, m_1) = \frac{m_1 R T_1}{lS_A - \bar{V}_1 + xS_c}. \quad (5)$$

The distance between the piston and the right-hand wall of a hydraulic cylinder is  $x_2 = H - h - x$ . Respectively, the equation of the balance of volumes (1) for the right-hand side of the first hydraulic cylinder may be written as  $y_2 S_A + (H - h - x)S_c = \bar{V}_2$ . We obtain, then,

$$y_2 = \frac{\bar{V}_2 - (H - h - x)S_c}{S_A}. \quad (6)$$

Taking into account (6), we determine the gas volume in the upper chamber of a hydraulic accumulator,

$$v_2 = lS_A - \bar{V}_2 + (H - h)S_c + S_c x. \quad (7)$$

Since  $v_2 > 0$ , formula (7) yields the rightmost position of a piston  $x < x_{rm} = [\bar{V}_2 - lS_A - (H - h)S_c]/S_c$ .

The Mendelev–Klaiperon equation for volume  $v_2$  has the form

$$p_2 v_2 = \frac{m_2}{M} R_\mu T_2. \quad (8)$$

From (8), we determine  $p_2$ ,

$$p_2(x, m_2) = \frac{m_2 R T_2}{lS_A - \bar{V}_2 + (H - h)S_c - S_c x}, \quad (9)$$

where  $R = R_\mu/M$  is the universal gas constant.

The expressions for the pressure values in the left- and right-hand chambers of the right-hand hydraulic cylinder may be derived similarly:

$$p_3 = (x, m_3) = \frac{m_3 R T_2}{lS_A - \bar{V}_3 + xS_c}, \quad p_4(x, m_4) = \frac{m_4 R T_1}{lS_A - \bar{V}_4 + (H - h)S_c - S_c x}. \quad (10)$$

Let us introduce the following notations:  $b_1 = lS_A - \bar{V}_1$  and  $b_2 = lS_A - \bar{V}_2 + (H - h)S_c$ .

Using these notations, the expressions for the pressure on the piston wall (5), (9), and (10) may be rewritten as

$$p_1 = \frac{m_1 R T_1}{b_1 + xS_c}, \quad p_2 = \frac{m_2 R T_2}{b_2 - xS_c}, \quad p_3 = \frac{m_3 R T_2}{b_1 + xS_c}, \quad p_4 = \frac{m_4 R T_1}{b_2 - xS_c}. \quad (11)$$

The resulting force acting on the rod with pistons is

$$F_{rc} = (p_1 - p_2 + p_3 - p_4)S_c. \quad (12)$$

The dynamic equation for the rod with pistons has the form

$$m_{rc} \ddot{x} = F_{rc}(x) + F_{rs}(x), \quad (13)$$

where  $F_{rs}(x)$  is the velocity-dependent resistance force and  $m_p = 2m_{ps} + m_{rd}$ .

Dynamic equation (13) should be complemented with the dynamic equations describing transport of the working medium's mass through regenerators

$$\frac{dm_1}{dt} = r(p_1 - p_2), \quad \frac{dm_4}{dt} = -r(p_4 - p_3) \quad (14)$$

and the mass-balance equation

$$m_1 + m_2 + m_d = m_{01}, \quad m_3 + m_4 + m_d = m_{02}. \quad (15)$$

Equations (13)–(15) form a closed system of nonlinear differential equations of the fourth order. It can be solved only by numerical methods (owing to nonlinearity), as is the case with the more simple models of external-heating engines [4, 5].

However, for such a model, it is reasonable to find an approximate solution for the dynamics that would enable analysis of a steady-state periodic process in a closed form. To develop such approximate model, we make the following assumptions: function  $F_p(x)$  is linearized with regard to the central position of the rod's displacement, i.e., the point  $x_c = (H - h)/2$ ; continuously changing gas masses in hydraulic accumulators  $m_1$ ,  $m_2$ ,  $m_3$ , and  $m_4$  are approximated with piecewise constant functions of the direct and inverse motion of a rod; and self-vibrations are assumed to be symmetric.

Owing to the second and third assumptions, the expression for pressure values (11) for the direct motion of a rod may be presented as

$$p_1(x) = \frac{a_1}{b_1 + xS_c}, \quad p_2(x) = \frac{a_2}{b_2 - xS_c}, \quad p_3(x) = \frac{a_3}{b_1 + xS_c}, \quad p_4(x) = \frac{a_4}{b_2 - xS_c},$$

where  $a_1 = m_1RT_1$ ,  $a_2 = m_2RT_2$ ,  $a_3 = m_1RT_2$ , and  $a_4 = m_2RT_1$ .

In accordance with (12), the resulting force may be written in this notation as

$$F_{pn}(x) = \left( \frac{a_1}{b_1 + xS_c} - \frac{a_2}{b_2 - xS_c} + \frac{a_3}{b_1 + xS_c} - \frac{a_4}{b_2 - xS_c} \right) S_c. \quad (16)$$

When the rod is located in the central position  $x_c = (H - h)/2$ , (16) has the form

$$\begin{aligned} F_{pn}\left(\frac{H-h}{2}\right) &= \left[ (a_1 + a_3) \left/ \left[ b_1 + \frac{H-h}{2} S_c \right] - (a_2 + a_4) \left/ \left[ b_2 - \frac{H-h}{2} S_c \right] \right. \right] S_c, \\ F'_{pn}\left(\frac{H-h}{2}\right) &= - \left[ (a_1 + a_3) \left/ \left( b_1 + \frac{H-h}{2} S_c \right)^2 + (a_2 + a_4) \left/ \left( b_2 - \frac{H-h}{2} S_c \right)^2 \right. \right] S_c^2. \end{aligned} \quad (17)$$

Linearization of the function  $F_p(x)$  taking into account (17) yields

$$F_p(x) = F_p(x_c) + (x - x_c) F'_p(x_c). \quad (18)$$

We assume that the resistance force  $F_{rs}$  behaves as the linear viscous-friction force  $F_{rs} = kx$ . Substituting linearized expression (18) instead of the  $F_p(x)$  force into (13), we arrive at a differential equation describing the rod's motion from left to right,  $m_p \ddot{x} + k\dot{x} - (x - x_c) F'_p(x_c) = F_p(x_c)$ .

The equation describing the rod's motion from the right to the left has a similar form. However, one may make an approximation that the term  $F_p(x_c)$ , while having the same absolute value, changes its sign to the opposite one.

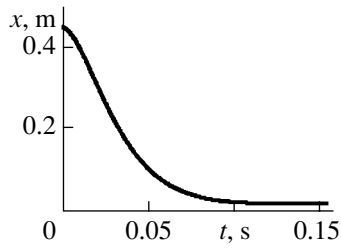


Fig. 2

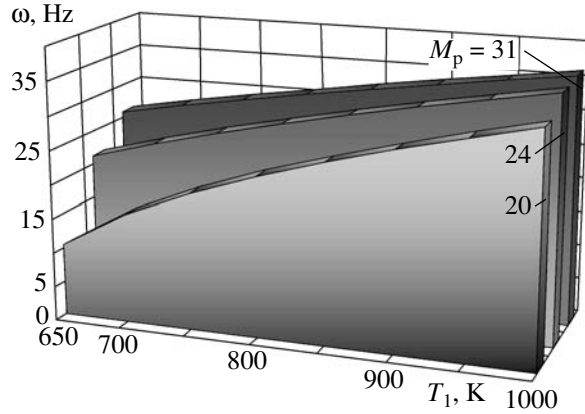


Fig. 3

Taking into account that  $F'_p(x_c)$  is negative in both cases and introducing displacement with respect to the rod's central position  $x_c$ ,  $x_1 = x - x_c$ , the equation for the rod's motion in both directions may be presented as

$$m_p \ddot{x} + k \dot{x}_1 + |F'_p(x_c)| x_1 = F_p(x_c) \operatorname{sgn} \dot{x}_1. \quad (19)$$

Equation (19) is identical to the known equation for self-vibrations of a system with negative dry friction [8], so that its rigorous analytical solution may be found by the fitting method. It is reasonable to start with its transformation to the form

$$\ddot{x}_1 + 2k \dot{x}_1 + \omega_0^2 x_1 = f_p(x_c) \operatorname{sgn} \dot{x}_1, \quad (20)$$

where  $2k = K/m_p$ ,  $\omega_0^2 = |F'_p(x_c)|/m_p$  and  $f_p = F_p(x_c)/m_p$ .

A solution to Eq. (20) in a steady-state periodic regime has the following form: for the motion from the right to the left ( $0 \leq t < T/2$ ),

$$x_1(t) = \left( x_1^* + \frac{f_p}{\omega_0^2} \right) e^{-kt} \left( \cos \omega t + \frac{k}{\omega} \sin \omega t \right) - \frac{f_p}{\omega_0^2}; \quad (21)$$

and for the motion from the left to the right ( $T/2 \leq t < T$ ),

$$x_1(t) = \left[ x_1 \left( \frac{T}{2} \right) - \frac{f_p}{\omega_0^2} \right] e^{-k \left( t - \frac{T}{2} \right)} \left\{ \cos \left[ \omega \left( t - \frac{T}{2} \right) \right] + \frac{k}{\omega} \sin \left[ \omega \left( t - \frac{T}{2} \right) \right] + \frac{f_p}{\omega_0^2} \right\}, \quad (22)$$

where  $\omega = \sqrt{\omega_0^2 - k^2}$  is the self-vibration frequency,  $T = 2\pi/\omega$  is the period of self-vibrations,

$x_1^* = (f_p/\omega_0^2) \left[ \left( 1 + e^{\frac{\pi k}{\omega}} \right) / \left( 1 - e^{\frac{\pi k}{\omega}} \right) \right]$  is the amplitude value of self-vibrations corresponding to the right-

most position of a rod, and  $x_1(T/2) = -x_1^* e^{\frac{\pi k}{\omega}} - (f_p/\omega_0^2) \left( 1 + e^{\frac{\pi k}{\omega}} \right)$  is the leftmost position of a rod (before the start of its motion to the right).

Figure 2 shows a plot of the rod's leftward motion in the steady-state regime  $x(t) = x_1(t) + x_c$  calculated according to (21) and (22) for the following values of parameters:  $H = 0.5$  m,  $h = 0.03$  m,  $l = 0.2$  m,  $S_A = 0.031$  m<sup>2</sup>,  $S_c = 0.126$  m<sup>2</sup>,  $\rho = 1.29$  kg/m<sup>3</sup> is the density of the working medium (air),  $R_u = 8.31$  J/(mol kg),  $M = 0.029$  kg/mol,  $m_{wm} = 0.021$  kg is the total mass of the working medium,  $m_d = 0.0084$  kg,  $T_1 = 750$  K is the heating temperature,  $T_2 = 295$  K is the cooling temperature,  $V_1 = V_4 = 0.0082$  m<sup>3</sup> is the first and the fourth total volume of the liquid,  $V_2 = V_3 = 0.126$  m<sup>3</sup> is the second and the third total volume of the liquid,  $m_{rd} = 10$  kg, and  $m_{ps} = 5$  kg.

Figure 3 shows the dependences of the vibration frequency  $\omega = \sqrt{|F'_p(x_c)|/m_p - (k^2/4m_p^2)}$  on the heating temperature  $T_1$  plotted for different values of the resulting mass  $m_p = 20, 24$ , and  $31$  kg.

#### ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 06-08-00755.

#### REFERENCES

1. Walker, G., *Stirling Engines*, Oxford: Clarendon Press, 1980.

2. *Dvigateli Stirlinga* (Stirling Engines), Kruglov, M.G., Ed., Moscow: Mashinostroenie, 1977.
3. *Dvigateli Stirlinga* (Stirling Engines), Brodyanskii, V.M., Ed., Moscow: Mir, 1975.
4. Sinev, A.V., Izrailovich, M.Ya., and Kangun, R.V., A Dynamic Model for the Self-Vibration Cycle of a Free-Piston External-Heating Engine, *Izv. Akad. Promyshl. Ekologii*, 2005, no. 3, pp. 37–42.
5. Kangun, R.V., A Mathematical Model for the Stirling Engine with a Crosshead Mechanism, *Izbran. trudy konf. MIKMUS-2005* (Selected Proc. of the MIKMUS-2005 Conf.), Moscow: IMASh RAN, 2006, pp. 80–84.
6. Shcherbakov, V.F., Maloshitskii, N.V., and Ponomarev, V.V., A Concept of an Environmentally Clean External-Combustion Engine, *Sb. nauchn. dokl. II Mezhd. sovv. po ispol'zovaniyu energoakkumuliruyushchikh veshchestv v ekologii, mashinostroenii, energetike* (Proc. II Int. Conf. on the Use of Energy Accumulating Substances in Ecology, Machine Building, and Power Engineering), Moscow: IMASh RAN, 2001, pp. 125–131.
7. Izrailovich, M.Ya., Removing Nonuniqueness and Stabilization of Periodic Regimes in Mechanical Systems, *Probl. Mashinostr. Nadezhnosti Mash.*, 2001, no. 4, pp. 25–30.
8. Panovko, Ya. G., *Vvedenie v teoriyu mekhanicheskikh kolebaniy* (Introduction to the Theory of Mechanical Vibrations), Moscow: Nauka, 1971.